ENERGY DISSIPATION IN UNSTEADY INCOMPRESSIBLE FLUID FLOWS

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We examine energy dissipation in the case of unsteady laminar flows of an incompressible viscous fluid. It is shown that depending on the motion conditions from 50 to 100% of the energy expended on tangential displacement of a solid surface in the fluid is dissipated. We also examine the question of evaluating the velocity fields found by solution of the approximate equations of motion of the fluid.

The following proposition holds: If solid boundaries whose displacement velocity thereafter remains constant are suddenly brought into tangential motion half of all the work expended is used to entrain into motion the viscous fluid enclosed within the moving boundaries in the absence of inertial accelerations; the second half of the work is dissipated into heat, and this ratio is independent of both the physical properties of the fluid and the boundary displacement velocity.

On the basis of the Helmholtz–Rayleigh comparison theorem we can state that for the same boundary motions but with the presence of inertial accelerations in the fluid the dissipated energy fraction exceeds one half.

We shall restrict ourselves to examination of only the most interesting particular cases.

1. For the case of motion of an infinite pipe along its axis, arising at the time t = 0 with the constant velocity U, we can obtain the following expression for the velocity of the fluid particles located inside the pipe:

$$u = U\left[1 - 2\sum_{k=1}^{\infty} \frac{J_0(\alpha_{0k} r/a)}{\alpha_{0k} J(\alpha_{0k})} \exp\left(-\frac{\alpha_{0k}^2}{a^2}\nu t\right)\right]$$

Here a = pipe radius, r = variable radius, J_0 = Bessel function of the first kind of zero order, α_{0k} = roots of the equation $J_0(\alpha) = 0$, ν = kinematic viscosity, t = time.

The specific (per unit pipe length) power dissipated in the fluid in the pipe motion is

$$D = 4\pi\mu U^2 \sum_{k=1}^{\infty} \exp\left(-2\frac{\alpha_{0k}^2}{a^2}\nu t\right)$$

and the specific power of the external forces expended on pipe motion is

$$N = 4\pi\mu U^2 \sum_{k=1}^{\infty} \exp\left(-\frac{\alpha_{0k}^2}{a^2}\nu t\right)$$

The ratio of the specific energy dissipated in the fluid by time t from initiation of the motion

$$A_d = 2\pi a^2 \rho U^2 \left[\frac{1}{4} - \sum_{k=1}^{\infty} \frac{1}{\alpha_{0k}^2} \exp\left(-2\frac{\alpha_{0k}^2}{a^2}\nu t\right)\right]$$

to the specific work of the external forces during this same time

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$$A = 4\pi a^2 \rho U^2 \left[\frac{1}{4} - \sum_{k=1}^{\infty} \frac{1}{\alpha_{0k}^2} \exp\left(- \frac{\alpha_{0k}^2}{a^2} \nu t \right) \right] \rightarrow \frac{1}{2} \quad \text{as} \quad t \rightarrow \infty$$

Thus, half the work expended is dissipated into heat, half is expended on entraining the fluid into motion.

2. For the case of rotation of an infinitely long cylinder of radius a about its axis, set into rotation at the time t = 0 with the constant angular velocity ω, we can obtain the following expression for the circumferential velocity of the fluid [1]

$$u = \omega a \left[\frac{r}{a} + 2 \sum_{k=1}^{\infty} \frac{J_1(\alpha_{1k} r/a)}{\alpha_{1k} J_0(\alpha_{1k})} \exp\left(- \frac{\alpha_{1k}^2}{a^2} \nu t \right) \right]$$

Here J_1 is the Bessel function of the first kind of first order, α_{1k} are the roots of the equation $J_1(\alpha) = 0$.

The rate of energy dissipation per unit fluid volume is

$$E = 4\mu\xi^2 - 4\mu \frac{\partial u}{\partial r} \frac{u}{r}, \quad \xi = \frac{1}{2}\left(\frac{u}{r} + \frac{\partial u}{\partial r} \right) \ (\xi = \text{vorticity})$$

The power dissipated in the fluid (per unit cylinder length) is

$$D = 4\pi\mu\omega^2 a^2 \sum_{k=1}^{\infty} \exp\left(-2 \frac{\alpha_{1k}^2}{a^2} \nu t \right)$$

The power supplied is

$$N = 4\pi\mu\, \omega^2 a^2 \sum_{k=1}^{\infty} \exp\left(- \frac{\alpha_{1k}^2}{a^2} \nu t \right)$$

And in this case the ratio of the specific energy dissipated

$$A_d = 2\pi\rho\, \omega^2 a^4 \left[\frac{1}{4} - \sum_{k=1}^{\infty} \frac{1}{\alpha_{1k}^2} \exp\left(-2 \frac{\alpha_{1k}^2}{a^2} \nu t \right) \right]$$

to the specific work of the external forces

$$A = 4\pi\rho\omega^2 a^4 \left[\frac{1}{8} - \sum_{k=1}^{\infty} \frac{1}{\alpha_{1k}^2} \exp\left(- \frac{\alpha_{1k}^2}{a^2} \nu t \right) \right] \rightarrow \frac{1}{2} \quad \text{as} \quad t \rightarrow \infty$$

On the basis of the Helmholtz theorem this ratio for the cylinder of finite length, and also for the ellipsoid, sphere, and so on, will exceed 1/2, and the greater the boundary velocity the greater is the dissipated energy fraction, other conditions being the same.

3. The characteristic feature of the problem is the condition that the fluid be bounded by moving surfaces.

If a solid surface performs tangential motion in an infinite fluid, then the dissipated energy fraction increases. Thus, for example, for the case of tangential motion in an infinitely deep liquid of an infinite plane which begins motion with the constant velocity U at the time t = 0 we can obtain the following expression for the fluid particle velocity [1]:

$$u = U \left(1 - \frac{2}{\pi} \int_0^{\infty} \exp(-\alpha^2 t) \sin \frac{\alpha y}{\sqrt{\nu}} \frac{d\alpha}{\alpha} \right)$$

where y is the ordinate, measured from the moving plane.

The power dissipated per unit area of the moving plane is

$$D = \rho U^2 \sqrt{\nu / 2\pi t}$$

the specific power of the external forces expended on motion of the plane is

$$N = \rho U^2 \sqrt{\nu / \pi t}$$

The ratio of the specific (per unit area) energy dissipated in the fluid

$$A_d = \rho U^2 \sqrt{2\nu t / \pi}$$

to the work expended

$$A = 2\rho U^2 \sqrt{\nu t / \pi}$$

equals in this case $1/\sqrt{2}$ and is independent of time.

Thus, in the case of tangential motion of a plane in an infinite fluid less than 30% of the work of the external forces is expended on entraining the fluid into motion and more than 70% of the work expended is dissipated.

At the same time, for the case of parallel motion of two planes at the distance h from one another and beginning motion at the time t = 0 in the same direction with the constant velocity U the ratio of the energy dissipated to the work expended equals 1/2, since in this case the fluid is enclosed between the moving surfaces.

In fact, it can be shown that the fluid velocity in this case is defined by the expression

$$u = U \left\{ 1 - \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k+1} \exp\left[- \frac{(2k+1)^2 \pi^2}{h^2} \nu t \right] \right\},$$

and the external force power expended on motion of the plates is

$$N = 8 \frac{\mu U^2}{h} \sum_{k=1}^{\infty} \exp\left[- \frac{(2k+1)^2 \pi^2}{h^2} \nu t \right],$$

and the power dissipated in the fluid is

$$D = 8 \frac{\mu U^2}{h} \sum_{k=1}^{\infty} \exp\left[-2 \frac{(2k+1)^2 \pi^2}{h^2} \nu t \right]$$

The ratio of the specific energy dissipated in the fluid during the time t

$$A_d = 4\rho \frac{U^2}{\pi^2} h \left\{ \frac{\pi^2}{8} - 1 - \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \exp\left[-2 \frac{(2k+1)^2 \pi^2}{h^2} \nu t \right] \right\}$$

to the work expended during this same time

$$A = 8\rho \frac{U^2}{\pi^2} h \left\{ \frac{\pi^2}{8} - 1 - \sum_{k=1}^{\infty} \frac{1}{(2k+1)^2} \exp\left[- \frac{(2k+1)^2 \pi^2}{h^2} \nu t \right] \right\}$$

equals 1/2 as $t \to \infty$.

We note that the specific (per unit surface area) work of the external forces expended on tangential displacement of the surfaces bounding the fluid is finite as $t \to \infty$, while the specific work of the tangential displacement of any surfaces in an infinite fluid increases without limit as $t \to \infty$.

For tangential motions of solid surfaces, accompanied by energy dissipation in the steady-state regimes, for example for the cases of rotation of a cylinder or a plane in an infinite fluid, motion of a plane parallel to some stationary plane, motion of a pipe in a pipe, and so on, the relative fraction of energy dissipated during the transient process time (i.e., during the period of motion acceleration) equals unity. In this case the transient process is accompanied by energy dissipation which exceeds infinitely the energy expended on entraining the fluid into motion.

In conclusion we note that the solutions of such problems as the Slezkin problem [1] on submergence of a flat plate into a viscous fluid, the Targ problem [2] on immersion of a pipe, solutions of various problems on boundary layer development, and so on, obtained on the basis of the solution of the approximate equations of motion, yield energetically impossible velocity fields in the fluid, since the energy dissipation in such fields is infinite. The reason for this is the unsatisfactory form of the velocity profile at the leading edge of the solid walls, obtained in adopting the boundary conditions of unperturbed fluid flow at the leading edge of the wall. In this case the surface of maximal velocity gradient is convex in relation to the surface being immersed, being at the same time infinite at the leading edge. For example, in the case of immersion of a pipe the power dissipated in the fluid volume inside the immersed part of the pipe of length h is

$$D = 4\pi\mu \int_0^a \int_0^h \left(\frac{\partial u}{\partial x}\right)^2 r\, dr\, dx = -4\pi\rho U \nu^2 \sum_{k=1}^{\infty} \exp\left(-2\frac{\alpha_{0k}^2}{Ua^2} x\right)\Bigg|_0^h \to \infty$$

although the power expended on immersion of the tube is finite

$$N = \pi\rho U^3 a^2 \left[1 - 4\sum_{k=1}^{\infty} \frac{1}{\alpha_{0k}^2} \exp\left(-\frac{\nu\alpha_{0k}^2}{Ua^2} h\right)\right]$$

LITERATURE CITED

1. N. A. Slezkin, Dynamics of Viscous Incompressible Fluids [in Russian], Gostekhizdat, Moscow (1955).
2. S. M. Targ, Fundamental Problems of Laminar Flow Theory [in Russian], Gostekhizdat, Moscow-Leningrad (1951).